

De Finetti's dividend problem and impulse control for a two-dimensional insurance risk process

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Abstract. Consider two insurance companies (or two branches of the same company) that have the same claims and they divide premia in some specified proportions. We model the occurrence of claims according to a Poisson process. The ruin is achieved if the corresponding two-dimensional risk process first leave the positive quadrant. We will consider two scenarios of controlled process: refraction and the impulse control. In first one the dividends are payed out when two-dimensional risk process exit fixed region. In the second scenario whenever process hits horizontal line, the risk process is reduced by paying dividend to some fixed point in the positive quadrant and waits there for the first claim to arrive. In both models we calculate discounted cumulative dividend payments until the ruin time. This paper is an attempt at understanding the effect of dependencies of two portfolios on the the join optimal strategy of paying dividends. For the proportional reinsurance we can observe for example the interesting phenomenon that is dependence of choice of the optimal barrier on the initial reserves. This is a contrast to the one-dimensional Cramér-Lundberg model where the optimal choice of barrier among optimal barrier strategies is uniform for all initial reserves.

Keywords: dividend, two-dimensional risk process, proportional reinsurance

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1 Introduction

In collective risk theory the reserves process X of an insurance company is modeled as

$$X(t) = u + ct - S(t), \quad (1)$$

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where $u > 0$ denotes the initial reserve,

$$S(t) = \sum_{i=1}^{N_t} U_i \quad (2)$$

is a compound Poisson. We assume that U_i ($i = 1, 2, \dots$) are i.i.d. distributed claims (with the distribution function F). The arrival process is a homogeneous Poisson process N_t with intensity λ . The premium income is modeled by a constant premium density c and the net profit condition is then $c > \frac{\lambda}{\alpha}$, where $\frac{1}{\alpha} = \mathbb{E}[U_1] < \infty$.

Recently, several authors have studied extensions of classical risk theory towards a multidimensional reserves model (1) where $X(t)$, x , c and $S(t)$ are vectors, with possible dependence between the components of $S(t)$. Indeed, the assumption of independence of risks may easily fail, for example in the case of reinsurance, when incoming claims have an impact on both insuring companies at the same time. In general, one can also consider situations where each claim event might induce more than one type of claim in an umbrella policy (see Sundt [41]). For some recent papers considering dependent risks, see Dhaene and Goovaerts [18, 19], Goovaerts and Dhaene [24], Müller [36, 37], Denuit et al. [14], Ambagaspitiya [2], Dhaene and Denuit [17], Hu and Wu [27] and Chan et al. [12].

In this paper we consider a particular two-dimensional risk model in which two companies split the amount they pay out of each claim in fixed proportions (for simplicity we assume that they are equal), and receive premiums at rates c_1 and c_2 , respectively. That is,

$$\underline{X}(t) = (X_1(t), X_2(t)) = \left(u_1 + c_1 t - \sum_{i=1}^{N_t} U_i, u_2 + c_2 t - \sum_{i=1}^{N_t} U_i \right). \quad (3)$$

Note that (1) models proportional reinsurance dependence. The same two-dimensional risk process was already considered in Avram et al. [6, 7, 8]. In Avram et al. [7] the asymptotics of the ruin probability when the initial reserves of both companies tend to infinity under a Cramér light-tail assumption on the claim size distribution is analyzed. In Avram et al. [6, 8] for the simplest particular case of exponential claims the Laplace transforms of few perpetual ruin probabilities are identified which allows to derive an explicit solution for above model. In all of these papers the dividend payments were not taken into account.

We shall assume that the second company, to be called reinsurer, receives less premium per amount paid out, that is,

$$c_1 > c_2. \quad (4)$$

Under the assumption made also in this paper that the premiums income per unit time c_i are larger than the average amount claimed $\lambda \mathbb{E}[U_1]$ the two-dimensional surplus in the model (3) has the unrealistic property that it converges to infinity on each co-ordinate with probability one. In answer to this objection De Finetti [20] introduced the dividend barrier model for one-dimensional

model (1), in which all surpluses above a given level are transferred to a beneficiary. Further, usually the payment of dividends should be made in such a way as to optimize the expected net present value of the total income of the shareholders from time zero until ruin. Associated to each dividend payment could be a fixed cost of size $K > 0$. Then it is no longer feasible to pay out dividends at a certain rate. Therefore the impulse control is considered instead where a dividend strategy is given by the pairs (T_i, J_i) for the stopping times T_i representing the times at which a dividend payment is made and for J_i being positive random variables representing the sizes of the dividend payments.

In the mathematical finance and actuarial literature there is a good deal of work on dividend barrier models and the problem of finding an optimal policy for paying out dividends. Gerber and Shiu [23], Grandits et al. [26] and Jeanblanc and Shiryaev [29] consider the optimal dividend problem in a Brownian setting. Irbäck [28], Zhou [43], Zajic [42], Avram et al. [5], Kyprianou and Palmowski [31] and Loeffen [33] study the constant barrier model for a classical and spectrally negative Lévy risk process. Azcue and Muler [10] follow a viscosity approach to investigate optimal reinsurance and dividend policies in the Cramér-Lundberg model. Since under De Finetti's objective ruin is sure and its "severity" ignored, several alternative objectives have been proposed recently, involving a penalty at ruin, based on a function of the severity of ruin. This kind of problems is considered in Loeffen and Renaud [35].

The impulse control literature is also vast. An important type of strategy for the impulse control problems is so-called (a^+, a^-) policy which is similar to the well known (s, S) policy appearing in inventory control models. The (a^-, a^+) policy is the strategy where each time the reserves are above a certain level a^+ , a dividend payment is made which brings the reserves down to another level a^- and where no dividends are paid out when the reserves are below a^- . Similar model of paying impulse dividends is considered paper where regulation of the risk process brings it to some fixed point. In case when risk process is a Brownian motion plus drift, Jeanblanc and Shiryaev [29] shows that an optimal strategy for the impulse control problem is formed by a (a^-, a^+) policy. Paulsen [38] considers the case when risk process is modeled by a diffusion process and showed that under certain conditions a (a^-, a^+) policy is optimal. Note that in Paulsen [38] this type of strategy is referred to as a lump sum dividend barrier strategy. Similar model is considered in Alvarez [1], Cadenillas et al. [13]. Loeffen [34] models reserves by a spectrally negative Lévy process and find conditions under which the barrier strategy is optimal. Avram et al. [9] find optimal impulse strategy for a Lévy risk process and polynomial penalty function.

We will follow the same arguments in the two-dimensional set-up introducing the following controlled risk process:

$$\underline{Y}(t) = (Y_1(t), Y_2(t)) = \underline{X}(t) - \underline{L}(t), \quad (5)$$

where $\underline{L}(t) = (L_1(t), L_2(t))$ and $L_i(t)$ ($i = 1, 2$) are nondecreasing \mathcal{F}_t -adaptable processes for a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ of $S(t)$.

In this paper analogously to the one-dimensional case we will consider two controlling mechanism: refraction and impulse payments. In Section 2 we deal with the refracted process \underline{Y} for which:

$$\underline{L}(t) = \left(\delta_1 \int_0^t \mathbf{1}_{\{\underline{Y}(t) \in \mathcal{B}\}}, \delta_2 \int_0^t \mathbf{1}_{\{\underline{Y}(t) \in \mathcal{B}\}} \right) \quad (6)$$

describes two-dimensional linear drift at rate

$$\underline{\delta} = (\delta_1, \delta_2) > (0, 0) \quad (7)$$

which is subtracted from the increments of the risk process whenever it enter the fixed set:

$$\mathcal{B} = \{(x, y) : x, y \geq 0 \text{ and } y \geq b - ax\}, \quad a, b > 0. \quad (8)$$

The case $\underline{\delta} = \underline{c} - \underline{a}$ for $\underline{c} = (c_1, c_2)$ and $\underline{a} = (-1, a)$ corresponds to the reflecting the risk process at the line $y = b - ax$. Then the risk process starting in $\mathcal{B}^c \cup \mathcal{B}_0$, where $\mathcal{B}_0 = \{(x, y) : x, y \geq 0 \text{ and } y = b - ax\}$, stays there up to the ruin time (see Figure 1). We will also assume that the first insurance company (that one with lesser initial reserves) will get eventually ruined, that is:

$$c_1 - \delta_1 < 0. \quad (9)$$

Note that this approach takes into account the dependence between the reserves of two companies in a procedure of paying the dividends. In particular, we allow of moving some reserves from one company to another in the form of dividend payment (or moving reserves between two branches of one insurance company). Besides it, in the case of independent risks the barrier is usually a quadrant in contrast to the model considered here where a line barrier forces true joint dividend policy of two insurance companies which are related with each other via proportional reinsurance.

In the Section 4 we consider impulse controlling where $L_i(t)$ ($i = 1, 2$) are cumulative payments made whenever two-dimensional risk process hits horizontal line $y = u_2$. The size of the i th payment equals $J_i = X_1(T_i) - u_1 + c_1 e_\lambda^{(i)}$ where T_i is the i th moment of hitting line $y = u_2$ by the process $\underline{X}(t)$ and $e_\lambda^{(i)}$ is independent of X exponential random variable with intensity λ . Each payment corresponds to reducing reserves to some fixed levels (u_1, u_2) and paying dividends while waiting for the next claim to arrive. Moreover, by condition (4) each time the second company (or first branch of company) has at least u_2 of reserves, the first company's reserves are greater than u_1 . The payments of the dividends are always made by the first company which has greater premium rate (by reducing the reserves to level u_1). Associated to each dividend payment is a fixed cost of size $K > 0$. In this model the reserves of the second branch of the insurance company serves as a random control mechanism giving us information when we should pay dividends from the reserves of the first branch. It is very convenient policy of paying dividend for insurance companies that have branches related via proportional reinsurance.

In this paper we focus on finding the n th moment

$$V_n(u_1, u_2) = V_n(\underline{u}) = \mathbb{E}[D^n | \underline{X}(0) = \underline{u}] \quad (10)$$

of the discount cumulative dividend payments

$$D = (1, 1) \cdot \int_0^\sigma e^{-qt} d\underline{L}(t), \quad (11)$$

made until the ruin time $\sigma = \inf \{t > 0 : \min(Y_1(t), Y_2(t)) < 0\}$ where q is a discounted factor. Above $(\alpha_1, \alpha_2) \cdot (\alpha_3, \alpha_4) = \alpha_1\alpha_3 + \alpha_2\alpha_4$ denotes the scalar product of two vectors in the plane. Note that σ is a first time when the controlled risk process exit the positive quadrant and it corresponds to the first time when at least one company get ruined.

The paper is organized as follows. In Section 2 we derive a partial integro-differential for V_n . In Section 3 we provide with explicit unique solution in terms of an infinite series in the case when claim sizes U_i are exponentially distributed. Finally, in Section 4 we solve the second dividend problem with the impulse control. Section 5 includes conclusions.

2 Barrier control

In this section for the risk process (3) we consider refracted process (5)-(6) for the barrier set (8). In this section we will derive partial integro-differential equation for V_n defined in (10).

Denote $\underline{u} = (u_1, u_2)$, $\delta_0 = \delta_1 + \delta_2$ and

$$\frac{\partial f}{\partial \underline{u}}(\underline{u}) = \left(\frac{\partial f}{\partial u_1}(\underline{u}), \frac{\partial f}{\partial u_2}(\underline{u}) \right)$$

for a general function f .

Theorem 1 *The partial derivatives $\frac{\partial V_n}{\partial u_i}(\underline{u})$ ($i = 1, 2$) exist and for $\underline{u} \in \mathcal{B}^c$ the function V_n is a unique solution of the equation:*

$$\underline{c} \cdot \frac{\partial V_n}{\partial \underline{u}}(\underline{u}) - (\lambda + nq)V_n(\underline{u}) + \lambda \int_0^{\min(u_1, u_2)} V_n(\underline{u} - (1, 1)v) dF(v) = 0 \quad (12)$$

with the boundary conditions:

$$n\delta_0 V_{n-1}(\underline{u}) = \underline{\delta} \cdot \frac{\partial V_n}{\partial \underline{u}} \Big|_{\underline{u} \in \mathcal{B}}, \quad \underline{u} \in \mathcal{B}, \quad (13)$$

$$\lim_{b \rightarrow \infty} V_n(\underline{u}) = 0, \quad \underline{u} \in \mathcal{B}^c, \quad (14)$$

$$V_n(0, b) = 0. \quad (15)$$

Proof For $y \geq 0$ define the Laplace transform:

$$M(\underline{u}, y) = \mathbb{E} [e^{-yD} | \underline{X}(0) = \underline{u}].$$

By the Strong Markov Property we have that for $\underline{u} \in \mathcal{B}^c$:

$$\begin{aligned} M(\underline{u}, y) &= (1 - \lambda dt) M(\underline{u} + \underline{c} dt, ye^{-qdt}) \\ &\quad + \lambda dt \int_0^{\min(u_1 + c_1 dt, u_2 + c_2 dt)} M(\underline{u} + \underline{c} dt - (1, 1)v, ye^{-qdt}) dF(v) \\ &\quad + \lambda dt \int_{\min(u_1 + c_1 dt, u_2 + c_2 dt)}^{\infty} dF(v) + o(dt). \end{aligned}$$

Then the Taylor expansion and collection of terms of order dt yields

$$\begin{aligned} \underline{c} \cdot \frac{\partial M}{\partial \underline{u}}(\underline{u}, y) - \lambda M(\underline{u}, y) - qy \frac{\partial M}{\partial y}(\underline{u}, y) + \lambda \int_0^{\min(u_1, u_2)} M(\underline{u} - (1, 1)v, y) dF(v) \\ + \lambda(1 - F(\min(u_1, u_2))) = 0. \end{aligned} \quad (16)$$

Moreover, for this equation we have the following boundary condition for $\underline{u} \in \mathcal{B}$:

$$\begin{aligned} M(\underline{u}, y) &= (1 - \lambda dt) e^{-y\delta_0 dt} M(\underline{u} + (\underline{c} - \underline{\delta}) dt, ye^{-qdt}) \\ &\quad + \lambda dt \int_0^{\min(u_1 + (c_1 - \delta_1) dt, u_2 + (c_2 - \delta_2) dt)} e^{-y\delta_0 dt} \\ &\quad \quad M(\underline{u} - (\underline{c} - \underline{\delta}) dt - (1, 1)v, ye^{-qdt}) dF(v) \\ &\quad + \lambda dt e^{-y\delta_0 dt} \int_{\min(u_1 + (c_1 - \delta_1) dt, u_2 + (c_2 - \delta_2) dt)}^{\infty} dF(v) + o(dt) \end{aligned}$$

which implies:

$$\begin{aligned} (\underline{c} - \underline{\delta}) \cdot \frac{\partial M}{\partial \underline{u}}(\underline{u}, y) \Big|_{\underline{u} \in \mathcal{B}} - (y\delta_0 + \lambda) M(\underline{u}, y) - qy \frac{\partial M}{\partial y}(\underline{u}, y) \Big|_{\underline{u} \in \mathcal{B}} \\ + \lambda \int_0^{\min(u_1, u_2)} M(\underline{u} - (1, 1)v, y) dF(v) + \lambda(1 - F(\min(u_1, u_2))) = 0. \end{aligned} \quad (17)$$

Note that $M(\underline{u}, y)$ and its partial derivatives with respect u_i ($i = 1, 2$) and y are continuous. Thus from (16) we finally derive for $\underline{u} \in \mathcal{B}$:

$$-y\delta_0 M(\underline{u}, y) = \underline{\delta} \frac{\partial M}{\partial \underline{u}}(\underline{u}, y) \Big|_{\underline{u} \in \mathcal{B}}. \quad (18)$$

Using representation $M(\underline{u}, y) = 1 + \sum_{n=1}^{\infty} \frac{(-y)^n}{n!} V_n(\underline{u})$ and equating the coefficient of $(-y)^n$ in (16) completes the proof of (12)-(13). Moreover, conditions (14)-(15) follows straightforward from definition of D in (11) and inequality (9).

We will prove the uniqueness of the solution using arguments similar like in Gerber [22]. It suffices to prove the uniqueness of the solution of (16) and (18).

For that purpose, we define an operator \mathcal{A} by

$$\begin{aligned} \mathcal{A}g(\underline{u}, y) = & \lambda \int_0^T e^{-\lambda t} \int_0^{\min(u_1+c_1t, u_2+c_2t)} g(\underline{u} + \underline{c}t - (1, 1)v, ye^{-qt}) dF(v) dt \\ & + \lambda E \int_T^\sigma e^{-\lambda t} \exp \left\{ -y\delta_0 \int_T^t e^{-qs} ds \right\} \int_0^{\min(u_1+(c_1-\delta_1)(t-T), u_2+(c_2-\delta_2)(t-T))} \\ & g(\underline{u} + (\underline{c} - \underline{\delta})(t - T) - (1, 1)v, ye^{-qt}) dF(v) dt, \end{aligned}$$

where $T = (b - au_1 - u_2)/(c_2 + ac_1)$ for $\underline{u} \in \mathcal{B}^c$ is a first time of getting to the linear barrier

$$\mathcal{B}_0 = \{(x, y) : x, y \geq 0 \text{ and } y = b - ax\}. \quad (19)$$

Note that in \mathcal{A} each increment can be interpreted as a conditioning on whether a claim occurs before the surplus process hits the barrier ($t < T$) or after this event (in which case we have an additional term $\exp \left\{ -y\delta_0 \int_T^t e^{-qs} ds \right\}$ representing the discounted dividends paid until the claim occurs). The solution M of (16) and (18) is a fixed point of the integral operator \mathcal{A} . For two functions g_1 and g_2 we have:

$$\begin{aligned} & |\mathcal{A}g_1(\underline{u}, y) - \mathcal{A}g_2(\underline{u}, y)| \\ & \leq \|g_1(\underline{u}, y) - g_2(\underline{u}, y)\|_\infty \left(\lambda \int_0^T e^{-\lambda t} dt \right. \\ & \quad \left. + \lambda \int_T^\infty e^{-\lambda t} \exp \left\{ -y\delta_0 \int_T^t e^{-qs} ds \right\} dt \right) \\ & < \|g_1(\underline{u}, y) - g_2(\underline{u}, y)\|_\infty, \end{aligned}$$

where $\|\cdot\|_\infty$ is the supremum norm over $\underline{u} \in \mathbb{R}^2$ and $y \in \mathbb{R}_+$. Thus it follows that \mathcal{A} is a contraction and the fixed point is unique by Banach's theorem. \square

Remark 1 Equation (16) (hence also equation (12)) could be also derived using Feynman-Kac formula.

3 An explicit solution for the exponential claims size and reflection

In this section we will find unique solution of (12)-(15) for $n = 1$, exponential claim size $F(v) = 1 - e^{-\alpha v}$ and reflecting at the line $y = b - ax$, that is $\underline{\delta} = \underline{c} - \underline{a}$, where $\underline{a} = (-1, a)$ (see Figure 1). Note that by (7) we have

$$c_2 > a. \quad (20)$$

Without loss of generality we will assume that $u_1 < u_2$. In the case when $u_1 \geq u_2$ the solution of (12)-(13) could be modified in the obvious way.

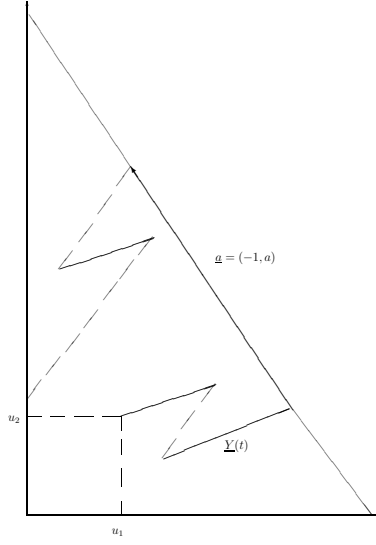


Figure 1: Controlled two-dimensional risk process.

Let

$$\gamma_{2,0}(m) = \frac{-[m(\alpha c_1 - q - \lambda) + \alpha c_2 - q - \lambda] + \sqrt{\Delta_{\gamma_{2,0}}(m)}}{2((m^2 + m)c_1 + c_2 + mc_2)} > 0, \quad (21)$$

where

$$\Delta_{\gamma_{2,0}}(m) = [m(\alpha c_1 - q - \lambda) + \alpha c_2 - q - \lambda]^2 + 4\alpha q[(m^2 + m)c_1 + c_2 + mc_2] > 0. \quad (22)$$

Moreover, for

$$a' = \frac{a - c_2}{c_1 + 1} < 0 \quad (23)$$

denote $\gamma_{2,0} = \gamma_{2,0}(a)$, $\gamma'_{2,0} = \gamma_{2,0}(a')$ and

$$\gamma_{1,0} = a\gamma_{2,0} > 0, \quad \gamma_{3,k} - a\gamma_{2,k} = \gamma_{1,k+1} - a\gamma_{2,k+1} \quad (24)$$

and

$$\gamma'_{1,0} = a'\gamma'_{2,0} > 0, \quad \gamma'_{3,k} - a'\gamma'_{2,k} = \gamma'_{1,k+1} - a'\gamma'_{2,k+1}. \quad (25)$$

We also introduce recursively coefficients:

$$D_0 = e^{-\gamma_{2,0}(a)b} \frac{(c_1 + 1) + (c_2 - a)}{\gamma_{1,0}(c_1 + 1) + \gamma_{2,0}(a)(c_2 - a)}; \quad (26)$$

$$D_{k+1} = \frac{\gamma_{3,k} + \gamma_{2,k} + \alpha}{\gamma_{1,k} + \gamma_{2,k} + \alpha} \cdot \frac{\gamma_{3,k}(c_1 + 1) + \gamma_{2,k}(c_2 - a)}{\gamma_{1,k+1}(c_1 + 1) + \gamma_{2,k+1}(c_2 - a)} D_k e^{\gamma_{2,k}b} e^{-\gamma_{2,k+1}b}. \quad (27)$$

Similarly, $D'_0 = 1$ and

$$D'_{k+1} = \frac{\gamma'_{3,k} + \gamma'_{2,k} + \alpha}{\gamma'_{1,k} + \gamma'_{2,k} + \alpha} \cdot \frac{\gamma'_{3,k}(c_1 + 1) + \gamma'_{2,k}(c_2 - a)}{\gamma'_{1,k+1}(c_1 + 1) + \gamma'_{2,k+1}(c_2 - a)} D'_k e^{\gamma'_{2,k}b} e^{-\gamma'_{2,k+1}b}. \quad (28)$$

Theorem 2 For $(u_1, u_2) \in \mathcal{B}^c$ we have:

$$V_1(u_1, u_2) = \sum_{k=0}^{\infty} D_k \left(e^{\gamma_{1,k}u_1} - \frac{\gamma_{3,k} + \gamma_{2,k} + \alpha}{\gamma_{1,k} + \gamma_{2,k} + \alpha} e^{\gamma_{3,k}u_1} \right) e^{\gamma_{2,k}u_2} \quad (29)$$

$$+ E \sum_{k=0}^{\infty} D'_k \left(e^{\gamma'_{1,k}u_1} - \frac{\gamma'_{3,k} + \gamma'_{2,k} + \alpha}{\gamma'_{1,k} + \gamma'_{2,k} + \alpha} e^{\gamma'_{3,k}u_1} \right) e^{\gamma'_{2,k}u_2}, \quad (30)$$

where

$$E = - \left(\sum_{k=0}^{\infty} D_k \frac{\gamma_{1,k} - \gamma_{3,k}}{\gamma_{1,k} + \gamma_{2,k} + \alpha} e^{\gamma_{2,k}b} \right) / \left(\sum_{k=0}^{\infty} D'_k \frac{\gamma'_{1,k} - \gamma'_{3,k}}{\gamma'_{1,k} + \gamma'_{2,k} + \alpha} e^{\gamma'_{2,k}b} \right). \quad (31)$$

Proof Taking $V_1 := V$ the equation (12) can be rewritten as follows:

$$c_1 \frac{\partial V}{\partial u_1} + c_2 \frac{\partial V}{\partial u_2} - (\lambda + q)V(\underline{u}) + \lambda \int_0^{\min(u_1, u_2)} V(\underline{u} - (1, 1)v) \alpha e^{-\alpha v} dv = 0. \quad (32)$$

Applying an operator $\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}$ to the equation (32) gives

$$\begin{aligned} c_1 \frac{\partial V}{\partial u_1^2} + c_2 \frac{\partial V}{\partial u_2^2} + (c_1 + c_2) \frac{\partial V}{\partial u_1 \partial u_2} + (\alpha c_1 - q - \lambda) \frac{\partial}{\partial u_1} V(\underline{u}) \\ + (\alpha c_2 - q - \lambda) \frac{\partial}{\partial u_2} V(\underline{u}) - \alpha q V(\underline{u}) = 0 \end{aligned} \quad (33)$$

since

$$\begin{aligned} \left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2} \right) \int_0^{\min(u_1, u_2)} V(u_1 - v, u_2 - v) \alpha e^{-\alpha v} dv \\ = -\alpha \int_0^{\min(u_1, u_2)} V(u_1 - v, u_2 - v) \alpha e^{-\alpha v} dv + \alpha V(\underline{u}). \end{aligned}$$

Since $V_0(\underline{u}) = 1$ the boundary condition (13) translates into

$$(c_1 + 1) + (c_2 - a) = (c_1 + 1) \frac{\partial V}{\partial u_1} \Big|_{\underline{u} \in \mathcal{B}_0} + (c_2 - a) \frac{\partial V}{\partial u_2} \Big|_{\underline{u} \in \mathcal{B}_0}, \quad (34)$$

where \mathcal{B}_0 is defined (19). We will look first for a solution of equation (33) in a following form:

$$(\Upsilon_1 e^{\gamma_1 u_1} + \Upsilon_2 e^{\gamma_3 u_1}) e^{\gamma_2 u_2} \quad (35)$$

for some constants Υ_1 and Υ_2 . Putting (35) into (33) shows that γ_1 and γ_3 are two real roots of the equation (later we prove that they always exist):

$$c_1 \gamma^2 + c_2 \gamma_2^2 + (c_1 + c_2) \gamma \gamma_2 + (\alpha c_1 - q - \lambda) \gamma + (\alpha c_2 - q - \lambda) \gamma_2 - \alpha q = 0. \quad (36)$$

Recall that $u_1 < u_2$. Then from (32) we obtain:

$$\begin{aligned} \Upsilon_1 \left(c_1 \gamma_1 + c_2 \gamma_2 - (\lambda + \delta) + \frac{\lambda \alpha}{\gamma_1 + \gamma_2 + \alpha} \right) e^{\gamma_1 u_1 + \gamma_2 u_2} \\ - \frac{\lambda \Upsilon_1 \alpha}{\gamma_1 + \gamma_2 + \alpha} e^{-(\gamma_2 + \alpha) u_1 + \gamma_2 u_2} + \Upsilon_2 \left(c_1 \gamma_3 + c_2 \gamma_2 - (\lambda + \delta) \right. \\ \left. + \frac{\lambda \alpha}{\gamma_3 + \gamma_2 + \alpha} \right) e^{\gamma_3 u_1 + \gamma_2 u_2} - \frac{\lambda \Upsilon_2 \alpha}{\gamma_3 + \gamma_2 + \alpha} e^{-(\gamma_2 + \alpha) u_1 + \gamma_2 u_2} = 0. \end{aligned}$$

Note that $c_1\gamma_1 + c_2\gamma_2 - (\lambda + \delta) + \frac{\lambda\alpha}{\gamma_1 + \gamma_2 + \alpha} = c_1\gamma_3 + c_2\gamma_2 - (\lambda + \delta) + \frac{\lambda\alpha}{\gamma_3 + \gamma_2 + \alpha} = 0$. Thus

$$\Upsilon_2 = \frac{-\Upsilon_1(\gamma_3 + \gamma_2 + \alpha)}{\gamma_1 + \gamma_2 + \alpha}$$

and (35) can be rewritten in the following form:

$$\Upsilon_1 \left(e^{\gamma_1 u_1} - \frac{\gamma_3 + \gamma_2 + \alpha}{\gamma_1 + \gamma_2 + \alpha} e^{\gamma_3 u_1} \right) e^{\gamma_2 u_2} \quad (37)$$

To fulfill boundary condition (34) we take $V = V_1$ which is given by (29)-(30). It is not a priori clear that the solution is of the above form, but if such a solution can be found, we will solve (33) by uniqueness demonstrated in Theorem 1.

We will choose $\gamma_{1,k}, \gamma_{3,k} < \gamma_{1,k}$ and $\gamma'_{1,k}, \gamma'_{3,k} < \gamma'_{1,k}$ that solve the following equations:

$$c_1\gamma^2 + c_2\gamma_{2,k}^2 + (c_1 + c_2)\gamma\gamma_{2,k} + (\alpha c_1 - q - \lambda)\gamma + (\alpha c_2 - q - \lambda)\gamma_{2,k} - \alpha q = 0 \quad (38)$$

and

$$c_1\gamma^2 + c_2\gamma_{2,k}^{2'} + (c_1 + c_2)\gamma\gamma'_{2,k} + (\alpha c_1 - q - \lambda)\gamma + (\alpha c_2 - q - \lambda)\gamma'_{2,k} - \alpha q = 0, \quad (39)$$

respectively. To satisfy the boundary condition (34) for V_1 we take

$$\begin{aligned} & \sum_{k=0}^{\infty} D_k e^{\gamma_{2,k}(b-au_1)} \left(e^{\gamma_{1,k}u_1} (\gamma_{1,k}(c_1 + 1) + \gamma_{2,k}(c_2 - a)) \right. \\ & \quad \left. - \frac{\gamma_{3,k} + \gamma_{2,k} + \alpha}{\gamma_{1,k} + \gamma_{2,k} + \alpha} (\gamma_{3,k}(c_1 + 1) + \gamma_{2,k}(c_2 - a)) e^{\gamma_{3,k}u_1} \right) \\ & = (c_1 + 1) + (c_2 - a) \end{aligned} \quad (40)$$

and

$$\begin{aligned} & \sum_{k=0}^{\infty} D'_k e^{\gamma'_{2,k}(b-au_1)} \left(e^{\gamma'_{1,k}u_1} (\gamma'_{1,k}(c_1 + 1) + \gamma'_{2,k}(c_2 - a)) \right. \\ & \quad \left. - \frac{\gamma'_{3,k} + \gamma'_{2,k} + \alpha}{\gamma'_{1,k} + \gamma'_{2,k} + \alpha} (\gamma'_{3,k}(c_1 + 1) + \gamma'_{2,k}(c_2 - a)) e^{\gamma'_{3,k}u_1} \right) \\ & = 0. \end{aligned} \quad (41)$$

Hence (24)-(28) holds. Note that from (14) it follows that $\gamma_{2,k} > 0$ and $\gamma'_{2,k} > 0$ ($k = 0, 1, \dots$). The coefficient E we choose in such way to satisfy last boundary condition (15). Note also that there will be no more functions being linear combination of functions of type (37) such that (40) and (41) will be satisfied since all coefficients are uniquely determined.

The following facts proved in the Appendix could be collected.

Lemma 3.1 *Let $k \in \mathbb{N} \cup \{0\}$. Then*

- (i) $\gamma_{2,0} > 0$ and $\gamma'_{2,0} > 0$;
- (ii) $\gamma_{i,k}$ and $\gamma'_{i,k}$ ($i = 1, 2, 3$, $k = 0, 1, 2, \dots$) satisfying (24), (38) and (25), (39), respectively, always exist;
- (iii) $\gamma_{2,k+1} > \gamma_{2,k} > 0$, $\gamma_{1,k} > \gamma_{3,k}$ and $\gamma_{3,k+1} < \gamma_{3,k} < 0$; similarly $\gamma'_{2,k+1} > \gamma'_{2,k} > 0$, $\gamma'_{1,k} > \gamma'_{3,k}$ and $\gamma'_{3,k+1} < \gamma'_{3,k} < 0$.

To find V one needs to calculate $\gamma_{1,0}, \gamma_{2,0}$ and D_0 first. Then based on $\gamma_{2,0}$ one should find the smallest negative solution $\gamma_{3,0}$ of the square equation (38). In the next recursive step using (24) by the representation $\gamma_{1,k+1} = \gamma_{3,k} - a\gamma_{2,k} + a\gamma_{2,k+1}$ one can identify the biggest positive $\gamma_{2,k+1}$ satisfying (38). Then (38) produces $\gamma_{1,k+1}$ and $\gamma_{3,k+1}$. The constants D_k are given recursively by (27). Similarly one can identify $\gamma'_{i,k}$ ($i = 1, 2, 3$, $k = 0, 1, 2, \dots$). In the last step we calculate the constant E .

We will prove that now that series in (29)-(30) are convergent. To do that we first give the following properties which will be proved in the Appendix.

Lemma 3.2 *Let $k \in \mathbb{N} \cup \{0\}$. Then*

- (i) $\lim_{k \rightarrow \infty} \gamma_{2,k} = \lim_{k \rightarrow \infty} \gamma'_{2,k} = \infty$ and $\lim_{k \rightarrow \infty} \gamma_{i,k} = \lim_{k \rightarrow \infty} \gamma'_{i,k} = -\infty$ for $i = 1, 3$;
- (ii) $\lim_{k \rightarrow \infty} \frac{\gamma_{i,k+1}}{\gamma_{i,k}} = \lim_{k \rightarrow \infty} \frac{\gamma'_{i,k+1}}{\gamma'_{i,k}} = \frac{c_1 a + c_1}{c_1 a + c_2} > 1$ for $i = 1, 2, 3$;
- (iii) $\lim_{k \rightarrow \infty} \frac{\gamma_{3,k}}{\gamma_{2,k}} = \lim_{k \rightarrow \infty} \frac{\gamma'_{3,k}}{\gamma'_{2,k}} = -1$ and $\lim_{k \rightarrow \infty} \frac{\gamma_{1,k}}{\gamma_{2,k}} = \lim_{k \rightarrow \infty} \frac{\gamma'_{1,k}}{\gamma'_{2,k}} = -\frac{c_2}{c_1}$.

Using the d'Alembert criterion it suffices to prove now that the series $\sum_{k=0}^{\infty} D_k e^{\gamma_{2,k} u_2} e^{\gamma_{1,k} u_1}$ and $\sum_{k=0}^{\infty} D_k e^{\frac{\gamma_{3,k} + \gamma_{2,k} + \alpha}{\gamma_{1,k} + \gamma_{2,k} + \alpha} \gamma_{3,k} u_1}$ given in (29) are convergent. Convergence of series appearing in (30) we prove in the same way. Note that indeed,

$$\begin{aligned} & \frac{D_{k+1}}{D_k} e^{\gamma_{2,k+1} u_2} e^{-\gamma_{2,k} u_2} e^{\gamma_{1,k+1} u_1} e^{-\gamma_{1,k} u_1} \\ &= \frac{\gamma_{3,k} + \gamma_{2,k} + \alpha}{\gamma_{1,k} + \gamma_{2,k} + \alpha} \cdot \frac{\gamma_{3,k}(c_1 + 1) + \gamma_{2,k}(c_2 - a)}{\gamma_{1,k+1}(c_1 + 1) + \gamma_{2,k+1}(c_2 - a)} \\ & e^{(\gamma_{2,k+1} - \gamma_{2,k})(u_2 - b)} e^{(\gamma_{1,k+1} - \gamma_{1,k})u_1} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

since by Lemma 3.2 and by observation that $b > u_2$,

$$\lim_{k \rightarrow \infty} \frac{\gamma_{3,k} + \gamma_{2,k} + \alpha}{\gamma_{1,k} + \gamma_{2,k} + \alpha} = \lim_{k \rightarrow \infty} e^{(\gamma_{2,k+1} - \gamma_{2,k})(u_2 - b)} = \lim_{k \rightarrow \infty} e^{(\gamma_{1,k+1} - \gamma_{1,k})u_1} = 0$$

and

$$\lim_{k \rightarrow \infty} \frac{\gamma_{3,k}(c_1 + 1) + \gamma_{2,k}(c_2 - a)}{\gamma_{1,k+1}(c_1 + 1) + \gamma_{2,k+1}(c_2 - a)} = \text{const.}$$

Similarly,

$$\begin{aligned}
& \frac{D_{k+1}}{D_k} e^{\gamma_{2,k+1}u_2} e^{-\gamma_{2,k}u_2} e^{\gamma_{3,k+1}u_1} e^{-\gamma_{3,k}u_1} \\
&= \frac{\gamma_{3,k+1} + \gamma_{2,k+1} + \alpha}{\gamma_{1,k+1} + \gamma_{2,k+1} + \alpha} \cdot \frac{\gamma_{1,k} + \gamma_{2,k} + \alpha}{\gamma_{3,k} + \gamma_{2,k} + \alpha} \\
&= \frac{\gamma_{3,k+1} + \gamma_{2,k+1} + \alpha}{\gamma_{1,k+1} + \gamma_{2,k+1} + \alpha} \cdot \frac{\gamma_{3,k}(c_1 + 1) + \gamma_{2,k}(c_2 - a)}{\gamma_{1,k+1}(c_1 + 1) + \gamma_{2,k+1}(c_2 - a)} \\
& e^{(\gamma_{2,k+1} - \gamma_{2,k})(u_2 - b)} e^{(\gamma_{3,k+1} - \gamma_{3,k})u_1} \rightarrow 0 \quad \text{as } k \rightarrow \infty.
\end{aligned} \tag{42}$$

□

For the numerical analysis we assume $\alpha = 2$, $c_1 = 4$, $c_2 = 3$, $\lambda = 1$, $q = 0.1$. The values of expected dividend payments $V_1(u_1, u_2)$ for $u_1 = 1$, $u_2 = 2$ and $u_1 = 2$, $u_2 = 3$ depending on a and b are given in the Tables 1 and 2 below. Note that always there exists optimal choice of the linear barrier (choice of its upper left end $(0, b)$ and its slope a). This choice depends on the initial reserves (u_1, u_2) . For $(u_1, u_2) = (1, 2)$ the optimal barrier is determined by $b = 14$ and $a = 0.1$ and for $(u_1, u_2) = (2, 3)$ the optimal barrier is determined by $b = 15$ and $a = 0.1$. This is contrast to the one-dimensional case where the choice of the barrier is given only via the premium rate and the distribution of the arriving claims.

| a | b | | | | | |
|-----|-------|-------|-------|-------|-------|-------|
| | 6 | 8 | 14 | 15 | 20 | 28 |
| 0.1 | 19.85 | 27.20 | 34.95 | 34.93 | 32.48 | 25.89 |
| 0.2 | 16.33 | 24.31 | 33.82 | 34.19 | 33.32 | 28.03 |
| 0.5 | 11.76 | 17.74 | 28.98 | 30.01 | 32.54 | 31.21 |
| 1 | 7.22 | 11.40 | 21.35 | 22.59 | 27.17 | 30.07 |

Table 1: Expected value of the dividend payments depending on a and b for fixed $(u_1, u_2) = (1, 2)$.

| a | b | | | | | |
|-----|-------|-------|-------|-------|-------|-------|
| | 6 | 8 | 14 | 15 | 20 | 28 |
| 0.1 | 19.07 | 27.42 | 36.51 | 36.58 | 34.21 | 27.34 |
| 0.2 | 17.17 | 24.34 | 35.22 | 35.69 | 35.01 | 29.55 |
| 0.5 | 10.94 | 17.50 | 29.93 | 31.07 | 33.99 | 32.78 |
| 1 | 6.59 | 11.07 | 21.86 | 23.21 | 28.19 | 31.43 |

Table 2: Expected value of the dividend payments depending on a and b for fixed $(u_1, u_2) = (2, 3)$.

The Table 3 gives $V(u_1, u_2)$ for fixed $a = 0.9$ and $b = 1.8$ when $u_1 < u_2$ and $u_2 \leq b - au_1$. From this Table it follows for example that for a given slope a of the linear barrier it is optimal to locate around some line. In this case it is a line $u_2 = u_1$.

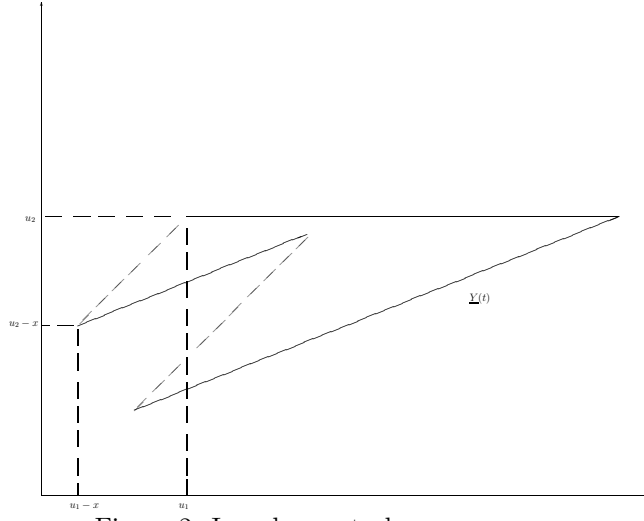


Figure 2: Impulse control.

| u_1 | u_2 | | | | | |
|-------|-------|------|------|------|------|------|
| | 0.2 | 0.4 | 0.6 | 0.8 | 0.9 | 1.2 |
| 0 | 2.09 | 1.58 | 1.11 | 0.69 | 0.49 | 0.03 |
| 0.1 | 2.35 | 1.81 | 1.31 | 0.86 | 0.65 | 0.13 |
| 0.2 | | 2.06 | 1.53 | 1.09 | 0.82 | 0.25 |
| 0.4 | | | 1.98 | 1.45 | 1.20 | 0.53 |
| 0.7 | | | | 2.11 | 1.83 | |
| 0.8 | | | | | 2.07 | |

Table 3: Expected value of the dividend payments depending on u_1 and u_2 for fixed a and b .

4 Impulse control

In this section we consider impulse controlling. The size of the i th payment made at epoch T_i equals $J_i = X_1(T_i) - u_1 + c_1 e_\lambda^{(i)}$ where T_i is the i th moment of hitting of $\underline{X}(t)$ in the horizontal line $y = u_2$ and $e_\lambda^{(i)}$ is independent of X exponential random variable with intensity λ . Each payment of the dividend corresponds to reducing reserves to some fixed levels (u_1, u_2) (see Figure 2) and paying out dividends while waiting for the next claim to arrive. By (4) at time T_i the first company has more than u_1 of reserves. Hence the payments of the dividends are always made by the first company which has greater premium rate (by reducing the reserves to the level u_1). Associated to each dividend payment is a fixed cost of size $K > 0$.

We will consider two cases, when $u_1 > u_2$ and when $u_1 \leq u_2$. In the first case the ruin can be only achieved by the second insurance company in contrast to the second case when both companies might get ruined.

4.1 Case $u_1 > u_2$

Define:

$$\begin{aligned}\tau_x^+ &= \inf\{t \geq 0 : c_2 t - S(t) = x\}, \\ \tau_{-u_2+x}^- &= \inf\{t \geq 0 : c_2 t - S(t) < -(u_2 - x)\}.\end{aligned}$$

Let x be a size of the first claim U_1 chosen according to the distribution function F . If there is no ruin at the moment of the arrival of the first claim then $x \leq u_2$. When the first claim arrives the risk process jumps then from (u_1, u_2) to $(u_1 - x, u_2 - x)$. Note that the first portion A of the payed dividends equals to the discounted independent exponential random variable $e_\lambda^{(1)} = e_\lambda$ with the parameter λ multiplied by c_1 plus discounted jump on the line $y = u_2$ which equals $(c_1 - c_2)\tau_x^+$ minus the costs of one payment made at the moment of impulse payment (that is at the moment of jump at the line $y = u_2$). Thus,

$$\begin{aligned} A &= c_1 \mathbb{E} \left[\int_0^{e_\lambda} e^{-qt} dt \right] \\ &\quad + \int_0^{u_2} \mathbb{E} \left[((c_1 - c_2)\tau_x^+ - K) e^{-q(\tau_x^+ + e_\lambda)} 1_{(\tau_x^+ < \tau_{-u_2+x}^-)} \right] dF(x) \\ &= \frac{c_1}{q + \lambda} + \frac{\lambda}{q + \lambda} \int_0^{u_2} \mathbb{E} \left[((c_1 - c_2)\tau_x^+ - K) e^{-q\tau_x^+} 1_{(\tau_x^+ < \tau_{-u_2+x}^-)} \right] dF(x). \end{aligned}$$

and mean of the cumulative discounted dividends equals:

$$\begin{aligned} V_1(u_1, u_2) &= A + A \underbrace{\int_0^{u_2} \mathbb{E}[e^{-q(e_\lambda + \tau_x^+)} 1_{(\tau_x^+ < \tau_{-u_2+x}^-)}] dF(x)}_p \\ &\quad + Ap^2 + Ap^3 + \dots = A \sum_{i=1}^{\infty} p^{i-1} = \frac{A}{1-p}, \end{aligned} \quad (43)$$

where

$$p = \frac{\lambda}{q + \lambda} \int_0^{u_2} \mathbb{E}[e^{-q\tau_x^+} 1_{(\tau_x^+ < \tau_{-u_2+x}^-)}] dF(x).$$

From Kyprianou [30] it follows that

$$\begin{aligned} \mathbb{E}[e^{-q\tau_x^+} 1_{(\tau_x^+ < \tau_{-u_2+x}^-)}] &= \frac{W^{(q)}(u_2 - x)}{W^{(q)}(u_2)}, \\ \mathbb{E}[\tau_x^+ e^{-q\tau_x^+} 1_{(\tau_x^+ < \tau_{-u_2+x}^-)}] &= -\frac{d}{dq} \left(\frac{W^{(q)}(u_2 - x)}{W^{(q)}(u_2)} \right), \end{aligned} \quad (44)$$

where $W^{(q)} : [0, \infty) \rightarrow [0, \infty)$ is a scale function, that is continuous and increasing function with the Laplace transform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = (\psi(\theta) - q)^{-1}, \quad \theta > \Phi(q), \quad (45)$$

for $\psi(\theta) = c_2\theta + \lambda(Ee^{-\theta U} - 1)$ being the Laplace exponent of X_1 and for $\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$ being its right inverse.

For the exponentially distributed claim sizes with parameter α we have that $\psi(\theta) = c_2\theta - \lambda\theta/(\alpha + \theta)$ and the scale function $W^{(q)}$ is given by

$$W^{(q)}(x) = c_2^{-1} \left(A_+ e^{q^+(q)x} - A_- e^{q^-(q)x} \right),$$

where $A_{\pm} = \frac{\alpha + q^{\pm}(q)}{q^{+}(q) - q^{-}(q)}$ and

$$q^{\pm}(q) = \frac{q + \lambda - \alpha c_2 \pm \sqrt{(q + \lambda - \alpha c_2)^2 + 4c_2 q \alpha}}{2c_2}.$$

Then,

$$p = \frac{\lambda \alpha}{c_2(q + \lambda)W^{(q)}(u_2)} \cdot \frac{e^{q^{+}(q)u_2} - e^{q^{-}(q)u_2}}{q^{+}(q) - q^{-}(q)}.$$

and

$$A = \frac{c_1}{q + \lambda} - Kp + \frac{\lambda}{q + \lambda}(c_1 - c_2) \int_0^{u_2} \mathbb{E} \left[\tau_x^{+} e^{-q\tau_x^{+}} 1_{(\tau_x^{+} < \tau_{-u_2+x}^{-})} \right] dF(x)$$

with

$$\begin{aligned} & \int_0^{u_2} \mathbb{E} \left[\tau_x^{+} e^{-q\tau_x^{+}} 1_{(\tau_x^{+} < \tau_{-u_2+x}^{-})} \right] dF(x) \\ &= \frac{\frac{d}{dq}W^{(q)}(u_2)}{(W^{(q)}(u_2))^2} \int_0^{u_2} W^{(q)}(u_2 - x) \alpha e^{-\alpha x} dx \\ & \quad - \frac{1}{W^{(q)}(u_2)} \int_0^{u_2} \frac{d}{dq}W^{(q)}(u_2 - x) \alpha e^{-\alpha x} dx \\ &= \frac{\frac{d}{dq}W^{(q)}(u_2)}{(W^{(q)}(u_2))^2} \frac{\alpha}{c_2} \frac{e^{q^{+}(q)u_2} - e^{q^{-}(q)u_2}}{q^{+}(q) - q^{-}(q)} \\ & \quad + \frac{1}{W^{(q)}(u_2)} \left[\frac{\alpha}{c_2(q^{+}(q) + \alpha)} A'_+ \left(e^{q^{+}(q)u_2} - e^{-\alpha u_2} \right) \right. \\ & \quad \left. - \frac{\alpha}{c_2(q^{-}(q) + \alpha)} A'_- \left(e^{q^{-}(q)u_2} - e^{-\alpha u_2} \right) \right. \\ & \quad \left. + \frac{\alpha}{c_2(q^{+}(q) + \alpha)} q^{+}(q)' A_+ \left(e^{q^{+}(q)u_2} - e^{-\alpha u_2} \right) \right. \\ & \quad \left. - \frac{\alpha}{c_2(q^{-}(q) + \alpha)} q^{-}(q)' A_- \left(e^{q^{-}(q)u_2} - e^{-\alpha u_2} \right) \right], \end{aligned}$$

where

$$\begin{aligned} q^{\pm}(q)' &= \frac{1}{2c_2} \left[1 \pm \frac{q + \lambda + \alpha c_2}{\sqrt{(q + \lambda - \alpha c_2)^2 + 4c_2 q \alpha}} \right], \\ A'_{\pm} &= \frac{d}{dq}A_{\pm} = \frac{q^{\pm}(q)'(q^{+}(q) - q^{-}(q)) - (\alpha + q^{\pm}(q))(q^{+}(q)' - q^{-}(q)')}{(q^{+}(q) - q^{-}(q))^2}, \\ \frac{d}{dq}W^{(q)}(x) &= c_2^{-1} \left(A'_+ e^{q^{+}(q)x} + q^{+}(q)' A_+ e^{q^{+}(q)x} - A'_- e^{q^{-}(q)x} - q^{-}(q)' A_- e^{q^{-}(q)x} \right). \end{aligned}$$

4.2 Case $u_1 \leq u_2$

Let x be a size of the first claim U_1 chosen according to the d.f. F . If there is no ruin at the moment of the arrival of the first claim then $x \leq u_1$. Define

$$\tau_U = \inf\{t \geq 0 : Z(t) = u_2\}$$

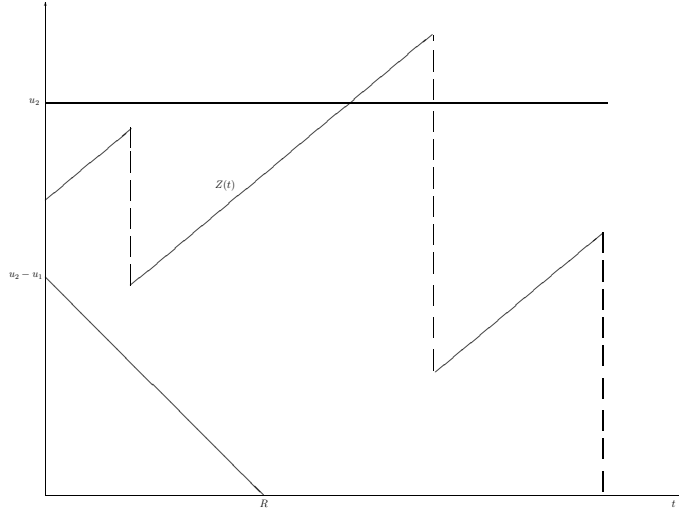


Figure 3: Piecewise lower barrier when $u_1 \leq u_2$.

$$\tau_L = \inf\{t \geq 0 : Z(t) < \max(0, u_2 - u_1 - (c_1 - c_2)t)\},$$

where $Z(t) = (u_2 - x) + c_2 t - S(t)$. Note that when $X_1(0) = u_1 - x$, $X_2(0) = Z(0) = u_2 - x \geq u_2 - u_1$, then $\{X_1(t) \geq 0, X_2(t) \geq 0\}$ is equivalent to the requirement that $\{\tau_L > t\}$ (see Figure 3). Moreover, the impulse payment equals $(c_1 - c_2)\tau_U$. Thus similarly like in the previous case:

$$V_1(u_1, u_2) = \frac{A}{1 - p}, \quad (46)$$

where

$$p = \frac{\lambda}{q + \lambda} \int_0^{u_2} \mathbb{E}[e^{-q\tau_U} 1_{(\tau_U < \tau_L)}] dF(x)$$

and

$$A = \frac{c_1}{q + \lambda} - Kp + \frac{\lambda(c_1 - c_2)}{q + \lambda} \int_0^{u_2} \mathbb{E}[\tau_U e^{-q\tau_U} 1_{(\tau_U < \tau_L)}] dF(x)$$

with

$$\mathbb{E}[\tau_U e^{-q\tau_U} 1_{(\tau_U < \tau_L)}] = -\frac{d}{dq} \mathbb{E}[e^{-q\tau_U} 1_{(\tau_U < \tau_L)}]. \quad (47)$$

To compute $\mathbb{E}[e^{-q\tau_U} 1_{(\tau_U < \tau_L)}]$ we introduce a new probability measure $P^{\Phi(q)}$:

$$\left. \frac{dP^{\Phi(q)}}{dP} \right|_{\mathcal{F}_t} = e^{\Phi(q)(Z(t) - Z(0)) - qt}.$$

On the new probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P^{\Phi(q)})$ we have $Z(t) = (u_2 - x) + c_2 t - S^{(q)}(t)$ for

$$S^{(q)}(t) = \sum_{i=1}^{N_t^{\Phi(q)}} U_i^{\Phi(q)},$$

and where $N_t^{\Phi(q)}$ is a Poisson process with intensity $\lambda_q = \lambda \tilde{F}(\Phi(q))$ for $\tilde{F}(\theta) = \int_0^\infty e^{-\theta x} F(dx)$ and $U^{\Phi(q)}$ has a distribution function:

$$F_q(dx) = e^{-\Phi(q)x} F(dx) / \tilde{F}(\Phi(q));$$

see Th. 4.8, p. 38 of Asmussen [4] and Rolski et al. [40] for details. Denote $\tau_i = \inf\{t \geq 0 : X_i(t) < 0\}$ ($i = 1, 2$).

Lemma 4.1 *We have,*

$$\mathbb{E} \left[e^{-q\tau_U} 1_{(\tau_U < \tau_L)} \right] = e^{-\Phi(q)x} \frac{V^{(q)}(u_2 - x)}{V^{(q)}(u_2)}, \quad (48)$$

where for $y \geq u_2 - u_1$ and $R = \frac{u_2 - u_1}{c_1 - c_2}$,

$$\begin{aligned} V^{(q)}(y) &= \int_0^{y+c_2R} P^{\Phi(q)}(\tau_1 > R, X_1(R) \in dz | X_1(0) = y - (u_2 - u_1)) \\ &\quad P^{\Phi(q)}(\tau_2 = \infty | X_2(0) = z) dz. \end{aligned} \quad (49)$$

Proof Note that from Optional Stopping Theorem

$$\mathbb{E} \left[e^{-q\tau_U} 1_{(\tau_U < \tau_L)} \right] = e^{-\Phi(q)x} P^{\Phi(q)}(\tau_U < \tau_L),$$

where we use fact that $Z(t \wedge \tau_U) \leq u_2$. Introduce $V^{(q)}(y) = P^{\Phi(q)}(\tau_L = \infty | Z(0) = y)$. Using strong Markov property

$$V^{(q)}(u_2 - x) = P^{\Phi(q)}(\tau_U < \tau_L) V^{(q)}(u_2),$$

since $P^{\Phi(q)}(\tau_U > \tau_L, \tau_L = \infty) = 0$. This completes the proof of (48). The identity (49) is also a consequence of Markov property applied to Z at time R being the zero of the line $y = (u_2 - u_1) - (c_1 - c_2)t$. \square

The terms appearing in (49) could be identified in the following way. Indeed, the ruin probability $P^{\Phi(q)}(\tau_2 < \infty | X_2(0) = z) = 1 - P^{\Phi(q)}(\tau_2 = \infty | X_2(0) = z)$ was analyzed in many papers. The reader is referred to the books by Gerber [21], Grandell [25], Asmussen [4] and Rolski et al. [40]. In particular, assume that the distribution function F is absolutely continuous with respect to the Lebesgue's measure. Define $g_j(x) = c_j \frac{dF_q(c_j x)}{dx}$ for $j = 1, 2$ and for $x > 0$,

$$f_t^j(x) = \frac{P^{\Phi(q)}(\frac{1}{c_j} S^{(q)}(t) \in dx)}{dx} = e^{-\lambda_q t} \sum_{i=1}^{\infty} \frac{(\lambda_q t)^i}{i!} g_j^{*i}(x) \text{ for } j = 1, 2. \quad (50)$$

From Beekman-Pollaczec-Khinchine formula we have

$$P^{\Phi(q)}(\tau_2 < \infty | X_2(0) = z) = \left(1 - \frac{\lambda_q}{c_2} \int_0^{\infty} y dF_q(y) \right) \int_{z/c_2}^{\infty} f_{y-z/c_2}^2(y) e^{\lambda(z/c_2 - y)} dy;$$

see e.g. Cor. 3.8 of Lefèvre and Loisel [32]. For the exponential claim size with parameter α , $\lambda_q = \lambda\alpha/(\alpha + \Phi(q))$ and $F_q(dx) = \alpha_q e^{-\alpha_q x} dx$ for $\alpha_q = \alpha + \Phi(q)$ and $\Phi(q) = \frac{(\lambda + q - c_2\alpha) + \sqrt{(\lambda + q - c_2\alpha)^2 + 4c_2q\alpha}}{2c_2}$. Moreover,

$$P^{\Phi(q)}(\tau_2 < \infty | X_2(0) = z) = \frac{\lambda_q}{c_2\alpha_q} \exp \{ -(\alpha_q - \lambda_q/c_2)z \};$$

see e.g. Cor. 3.2, p. 63 of Asmussen [4] and Th. 9.1, p. 108 of Asmussen [3].

Similarly, from the ballot theorem (see Borovkov [11], Picard and Lefèvre [39] and Lem. 3.3 of Lefèvre and Loisel [32]) we have the following lemma.

Lemma 4.2 For $\varphi(z) = v - \frac{z}{c_1} + R$ and $v = (y - (u_2 - u_1))/c_1$,

$$\begin{aligned} & \frac{1}{dz} P^{\Phi(q)}(\tau_1 > R, X_1(R) \in dz | X_1(0) = c_1 v) \\ &= f_R^1(\varphi(z)) - e^{-\lambda_q z / c_1} f_{R-z/c_1}^1(\varphi(z)) \\ & \quad - \int_v^{\varphi(z)} \frac{z}{c_1(R+v-w)} f_{R+v-w}^1(\varphi(z)-w) f_{w-v}^1(w) dw. \end{aligned} \quad (51)$$

To summarize, to find the cumulative dividend payments in this case we use identity (46) which is based on finding $\mathbb{E}[e^{-q\tau_U} 1_{(\tau_U < \tau_L)}]$ identified in Lemma 4.1 via $P^{\Phi(q)}(\tau_2 = \infty | X_2(0) = z)$ and $P^{\Phi(q)}(\tau_1 > R, X_1(R) \in dz | X_1(0) = c_1 v)$ for some v . This both quantities could be found using function f_t^j ($j = 1, 2$) given in (50). Unfortunately, more explicit expression for the expected value of the dividend payments is impossible to derive in contrast to the case when $u_1 > u_2$. Although De Vylder and Goovaerts [15, 16] show how to analyse numerically expressions of type (51), calculating expected dividend payments $V_1(u_1, u_2)$ for $u_1 < u_2$ still comes up against problems with identifying (47) and (49).

5 Conclusions

In this paper we analyzed the joint dividend payments for the two-dimensional risk process, where the reserves of two insurance companies are related with each other through the proportional reinsurance. We considered two control mechanisms: refracting at the linear barrier and the impulse control.

In the first case we derived partial differential equations for the n th moment of the cumulative dividend payments. We found also an explicit expression for the expected value of dividend payments when arriving claims have exponential distribution and risk process is reflected at the linear barrier. It appeared that in contrast to one-dimensional case this quantity is given by more complex expression being the linear combination of two series. Besides it, the numerical analysis showed that the optimal choice of the barrier (its upper left end $(0, b)$ and its slope a) depends on the initial reserves of both companies. Thus the dependence of two companies might affect the choice of the optimal strategy.

The impulse control with the positive transaction costs produces more complex expressions for the mean of the dividend payments. The expression for the case when the first company has less initial reserves than the second company appeared to be much more difficult to analyze than under the complementary condition. This is imposed by the way we pay dividends: the payments (reducing reserves to the fixed level) are always made by the first company and they are realized when the second company has more than another fixed amount of reserves. Thus reserves of the second company serves us a control mechanism in this case.

In future research of analyzing dividend problem for the two-dimensional risk process, it would be interesting to extend our analysis to the case of a more

sophisticated situation, such as introducing the penalty function which takes into account the severity of ruin, defining differently the ruin time or considering other types of the dependencies between reserves of the insurance companies. In particular, in our model the claims arrive to the both companies at the same time which induces jumps only in one direction. Relaxing dependence structure could be realized by allowing some claims to arrive to one company without making loss at the second company. In other words, it would be interesting to consider different directions of jumps. Nevertheless, we leave that point for the future research, since it seems that our methodology cannot be applied straightforward to that extended framework, and in the case of linear barrier it will result in more sophisticated partial differential equations.

Appendix

Proof of Lemma 3.1 Putting $\gamma_{1,0} = a\gamma_{2,0}$ into (38) gives equation:

$$\mathcal{I}_1 := ((a^2 + a)c_1 + (1 + a)c_2)\gamma_{2,0}^2 + (a(\alpha c_1 - q - \lambda) + \alpha c_2 - q - \lambda)\gamma_{2,0} - \alpha q = 0$$

which has one positive solution $\gamma_{2,0}$ given in (21) and one negative solution, since $\frac{-\alpha q}{(a^2 + a)c_1 + (1 + a)c_2} < 0$. Moreover, by (4) we have

$$(a^{2'} + a')c_1 + (1 + a')c_2 > 0 \quad (52)$$

and hence equation

$$((a^{2'} + a')c_1 + (1 + a')c_2)\gamma_{2,0}'^2 + (a'(\alpha c_1 - q - \lambda) + \alpha c_2 - q - \lambda)\gamma_{2,0}' - \alpha q = 0$$

has also one positive solution $\gamma_{2,0}'$. This completes the proof of (i). Note that $\gamma_{3,0}$ and $\gamma_{3,0}'$ are also well-defined since for (38):

$$\Delta_{\gamma_{1,0}} := (c_1 - c_2)^2\gamma_{2,0}^2 + 2\gamma_{2,0}(c_1 - c_2)(\alpha c_1 + \lambda + q) + (\alpha c_1 - \lambda - q)^2 + 4c_1\alpha q > 0$$

and

$$(c_1 - c_2)^2\gamma_{2,0}'^2 + 2\gamma_{2,0}'(c_1 - c_2)(\alpha c_1 + \lambda + q) + (\alpha c_1 - \lambda - q)^2 + 4c_1\alpha q > 0.$$

From (38) we have that

$$\mathcal{I}_2 := c_1\gamma_{1,0}\gamma_{3,0} = c_2\gamma_{2,0}^2 + (\alpha c_2 - q - \lambda)\gamma_{2,0} - \alpha q < 0, \quad (53)$$

where the last inequality follows from the form of \mathcal{I}_1 . Indeed, then

$$\mathcal{I}_2 = -\gamma_{2,0}[(a^2 + a)c_1 + ac_2]\gamma_{2,0} + a(\alpha c_1 - q - \lambda)$$

is negative since by (i) $\gamma_{2,0} > 0$ and by (21)

$$\begin{aligned} \gamma_{2,0} - \frac{-a(\alpha c_1 - q - \lambda)}{(a^2 + a)c_1 + ac_2} \\ = (c_1 - c_2)(a\alpha c_2 + \lambda a + qa + qa^2) + a(a^2 + a)c_1(\alpha c_1 - \lambda) \\ - qa(a^2 + a)c_1 + \sqrt{\Delta_{\gamma_{2,0}}}((a^2 + a)c_1 + ac_2) > 0. \end{aligned} \quad (54)$$

The inequality (54) is a consequence of the assumption (4), the net profit condition $c_1 > \frac{\lambda}{\alpha}$ and the inequality

$$\sqrt{\Delta_{\gamma_{2,0}}} > qa$$

which is by (22) equivalent to

$$\begin{aligned} \Delta_{\gamma_{2,0}} - q^2 a^2 &= (a\alpha c_1 - \lambda a + \alpha c_2 - q - \lambda)^2 \\ &+ 2qa(a\alpha c_1 + \alpha c_2 + \lambda a + \lambda + q + 2\alpha c_1) + 4\alpha qc_2 > 0. \end{aligned}$$

Then by (53),

$$\gamma_{3,0} < 0. \quad (55)$$

By (20) we have $a - c_2 \leq 0$ and hence $\gamma'_{3,0} < \gamma'_{1,0} \leq 0$.

Assume now that for some $k \in \mathbb{N} \cup \{0\}$ the quantities $\gamma_{2,k} > 0, \gamma_{3,k} < 0$ exist. The solutions of equation (38) then equals

$$\gamma_{3,k} = \frac{-(c_1 + c_2)\gamma_{2,k} + \lambda + q - \alpha c_1 - \sqrt{\Delta_{\gamma_{1,k}}}}{2c_1} \quad (56)$$

$$\gamma_{1,k} = \frac{-(c_1 + c_2)\gamma_{2,k} + \lambda + q - \alpha c_1 + \sqrt{\Delta_{\gamma_{1,k}}}}{2c_1}, \quad (57)$$

where

$$\Delta_{\gamma_{1,k}} = (c_1 - c_2)^2 \gamma_{2,k}^2 + 2\gamma_{2,k}(c_1 - c_2)(\alpha c_1 + \lambda + q) + (\alpha c_1 - \lambda - q)^2 + 4c_1 \alpha q.$$

Note that $\gamma_{1,k} > \gamma_{3,k}$ and that $\Delta_{\gamma_{1,k}} > 0$. Moreover, using (24) we find that $\gamma_{1,k+1} = \gamma_{3,k} - a\gamma_{2,k} + a\gamma_{2,k+1}$. Putting it into (38) produces the equation for $\gamma_{2,k+1}$:

$$\begin{aligned} &((a^2 + a)c_1 + (1 + a)c_2)\gamma_{2,k+1}^2 \\ &+ [(\gamma_{3,k} - a\gamma_{2,k})(2c_1 a + c_1 + c_2) - (\lambda + q)(1 + a) + \alpha(ac_1 + c_2)]\gamma_{2,k+1} \\ &+ (c_1(\gamma_{3,k} - a\gamma_{2,k})^2 + (c_1 \alpha - \lambda - q)(\gamma_{3,k} - a\gamma_{2,k}) - \alpha q) = 0 \end{aligned} \quad (58)$$

which has a solution since:

$$\begin{aligned} \Delta_{\gamma_{2,k+1}} &= [\alpha(ac_1 + c_2) - (\lambda + q)(1 + a)]^2 \\ &+ 4\alpha q((a^2 + a)c_1 + (1 + a)c_2) + (\gamma_{3,k} - a\gamma_{2,k})^2(c_1 - c_2)^2 \\ &+ 2(\gamma_{3,k} - a\gamma_{2,k})(c_2 - c_1)((\lambda + q)(1 + a) + \alpha(c_2 + ac_1)) > 0. \end{aligned} \quad (59)$$

Furthermore,

$$\begin{aligned} \gamma_{2,k+1} - \gamma_{2,k} &= \sqrt{\Delta_{\gamma_{2,k+1}}} + (2c_1 a + c_1 + c_2)\sqrt{\Delta_{\gamma_{1,k}}} \\ &+ \frac{(c_1 - c_2)^2}{2c_1}\gamma_{2,k} + \frac{(c_1 - c_2)}{2}(\alpha + \lambda + q) > 0. \end{aligned} \quad (60)$$

Hence $\gamma_{2,k+1} > 0$ and

$$\Delta_{\gamma_{1,k+1}} = (c_1 - c_2)^2 \gamma_{2,k+1}^2 + 2\gamma_{2,k+1}(c_1 - c_2)(\alpha c_1 + \lambda + q) + (\alpha c_1 - \lambda - q)^2 + 4c_1 \alpha q > 0$$

which means that there exist two solutions $\gamma_{1,k+1}, \gamma_{3,k+1} < \gamma_{1,k+1}$ of equation (38). Similarly, by (59) and (60), using the same arguments we can prove that $\gamma'_{2,k+1} > \gamma'_{2,k}$ and that there exist two solutions $\gamma'_{1,k+1}, \gamma'_{3,k+1} < \gamma'_{1,k+1}$ of equation (39). This completes the proof of (ii). To prove (iii) note that:

$$\gamma_{3,k+1} - \gamma_{3,k} = (c_1 + c_2)(\gamma_{2,k} - \gamma_{2,k+1}) + \sqrt{\Delta_{\gamma_{1,k}}} - \sqrt{\Delta_{\gamma_{1,k+1}}} < 0$$

since $\sqrt{\Delta_{\gamma_{1,k}}} - \sqrt{\Delta_{\gamma_{1,k+1}}} < 0$ which is equivalent to the inequality

$$(c_1 - c_2)^2(\gamma_{2,k} - \gamma_{2,k+1})(\gamma_{2,k} + \gamma_{2,k+1}) + 2(\gamma_{2,k} - \gamma_{2,k+1})(c_1 - c_2)(\alpha c_1 + \lambda + q) < 0$$

that follows from (60). In the same way we prove inequality $\gamma'_{3,k+1} < \gamma'_{3,k}$. \square

Proof of Lemma 3.2 Recall that by Lemma 3.1(iii) the sequence $\{\gamma_{2,k}, k = 0, 1, 2, \dots\}$ is monotone. Hence if it were not true that $\lim_{k \rightarrow \infty} \gamma_{2,k} = \infty$, then it would exist $C > 0$ such that $C = \lim_{k \rightarrow \infty} \gamma_{2,k}$. Then by (58),

$$\begin{aligned} \lim_{k \rightarrow \infty} (\gamma_{2,k+1} - \gamma_{2,k}) &= \lim_{k \rightarrow \infty} \left(\sqrt{\Delta_{\gamma_{2,k+1}}} + (2c_1 a + c_1 + c_2) \sqrt{\Delta_{\gamma_{1,k}}} \right) \\ &\quad + \frac{(c_1 - c_2)^2}{2c_1} C + \frac{(c_1 - c_2)}{2} (\alpha + \lambda + q) > 0. \end{aligned}$$

But if $\lim_{k \rightarrow \infty} \gamma_{2,k} = \text{const}$ then $\lim_{k \rightarrow \infty} (\gamma_{2,k+1} - \gamma_{2,k}) = 0$ which gives a contradiction. Note that by (56) we have that $\lim_{k \rightarrow \infty} \frac{\gamma_{3,k}}{\gamma_{2,k}} = -1$ and hence $\lim_{k \rightarrow \infty} \gamma_{3,k} = -\infty$ and by (58),

$$\lim_{k \rightarrow \infty} \frac{\gamma_{2,k+1}}{\gamma_{2,k}} = \frac{ac_1 + c_1}{ac_1 + c_2} > 1. \quad (61)$$

Similarly, by (57) $\lim_{k \rightarrow \infty} \frac{\gamma_{1,k}}{\gamma_{2,k}} \rightarrow -\frac{c_2}{c_1}$, hence $\lim_{k \rightarrow \infty} \gamma_{1,k} = -\infty$. Moreover, $\lim_{k \rightarrow \infty} \frac{\gamma_{1,k+1}}{\gamma_{2,k}} = -\frac{c_2}{c_1} \frac{ac_1 + c_1}{ac_1 + c_2}$ and thus $\lim_{k \rightarrow \infty} \frac{\gamma_{1,k+1}}{\gamma_{1,k}} = \frac{c_1 a + c_1}{c_1 a + c_2}$. Similarly we can prove that $\lim_{k \rightarrow \infty} \frac{\gamma_{3,k+1}}{\gamma_{3,k}} = \frac{c_1 a + c_1}{c_1 a + c_2}$. The same limits we can obtain by exchanging $\gamma_{i,k}$ by $\gamma'_{i,k}$ for $i = 1, 2, 3$. \square

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References

- [1] Alvarez, L. (2004) A Class of Solvable Impulse Control Problems, *Appl. Math. Optim.* **49**, 265-295.
- [2] Ambagaspitiya, R.S. (1999) On the distributions of two classes of correlated aggregate claims. *Insurance Math. Econom.* **24**, 301-308.
- [3] Asmussen, S. (2003) *Applied Probability*, Wiley.
- [4] Asmussen, S. (2000) *Ruin Probabilities*, World Scientific, Singapore.
- [5] Avram, F., Palmowski, Z. and Pistorius, M. (2008) On the optimal dividend problem for a spectrally negative Lévy process, *Ann. Appl. Probab.* **17**(1), 156-180.
- [6] Avram, F., Palmowski, Z. and Pistorius, M. (2005) A two-dimensional ruin problem on the positive quadrant, with exponential claims: Feynman-Kac formula, Laplace transform and its inversion, *Monografías del Seminario Matemático García de Galdeano* **33**, 199-205.
- [7] Avram, F., Palmowski, Z. and Pistorius, M. (2008) Exit problem of a two-dimensional risk process from a cone: exact and asymptotic results, *Annals of Applied Probability* **18**(6), 2421-2449.
- [8] Avram, F., Palmowski, Z. and Pistorius, M. (2008) A two-dimensional ruin problem on the positive quadrant, *Insurance: Mathematics and Economics* **42**(1), 227-234.
- [9] Avram, F., Palmowski, Z. and Pistorius, M. (2009) Impulse control problem for a Lévy risk-process in the presence of a Gerber-Shiu penalty function, *Manuscript*.
- [10] Azcue, P. and Muler, N. (2005) Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model, *Mathematical Finance* **15**, 261-308.
- [11] Borovkov, A. (1976) *Stochastic processes in queueing theory*, Springer-Verlag.
- [12] Chan, W., Yang, H. and Zhang, L. (2003) Some results on the ruin probabilities in a two-dimensional risk model, *Insurance Math. Econom.* **32**, 345-358.
- [13] Cadenillas, A., Choulli, T., Taksar, M. and Zhang, L. (2006) Classical and impulse stochastic control for the optimization of the dividend and risk policies of an insurance firm, *Mathematical Finance* **16**(1), 181-202.
- [14] Denuit, M., Genest, C. and Marceau, E. (1999) Stochastic bounds on sums of dependent risks, *Insurance Math. Econom.* **25**, 85-104.

- [15] De Vylder, F. E. and Goovaerts, M. J. (1998) Recursive calculation of finite-time ruin probabilities, *Insurance: Mathematics and Economics* **7**, 1–7.
- [16] De Vylder, F. E. and Goovaerts, M. J. (1999) Explicit finite-time and infinite-time ruin probabilities in the continuous case, *Insurance: Mathematics and Economics* **24**, 155–172.
- [17] Dhaene, J. and Denuit, M. (1999) The safest dependence structure among risks, *Insurance Math. Econom.* **25**, 11–21.
- [18] Dhaene, J. and Goovaerts, M.J. (1996) Dependency of risks and stop-loss order, *ASTIN Bulletin* **26**, 201–212.
- [19] Dhaene, J. and Goovaerts, M.J. (1997) On the dependency of risks in the individual life model, *Insurance Math. Econom.* **19**, 243–253.
- [20] De Finetti, B. (1957) Su un’ipostazione alternativa dell teoria collettiva del rischio, *Trans. XV Intern. Congress Act.* **2**, 433–443.
- [21] Gerber, H. U. (1979) *An Introduction to Mathematical Risk Theory*, S. S. Huebner Foundation Monograph, University of Philadelphia, Philadelphia.
- [22] Gerber, H.U. (1981) On the probability of ruin in the presence of a linear dividend barrier, *Scandinavian Actuarial Journal*, 105–115.
- [23] Gerber, H.U. and Shiu, E.S.W. (2004) Optimal dividends: analysis with Brownian motion, *North American Actuarial Journal* **8**, 1–20.
- [24] Goovaerts, M.J. and Dhaene, J. (1996) The compound Poisson approximation for a portfolio of dependent risks, *Insurance Math. Econom.* **18**, 81–85.
- [25] Grandell, J. (1990) *Aspects of Risk theory*, Springer-Verlag, New York.
- [26] Grandits, F. Hubalek, W. Schachermayer, M. Zigo (2007) Optimal expected exponential utility of dividend payments in Brownian risk model, *Scandinavian Actuarial Journal* **2**, 73–107.
- [27] Hu, T. and Wu, Z. (1999) On the dependence of risks and the stop-loss premiums, *Insurance Math. Econom.* **24**, 323–332.
- [28] Irbäck, J. (2003) Asymptotic theory for a risk process with a high dividend barrier, *Scand. Actuarial J.* **2**, 97–118.
- [29] Jeanblanc, M. and Shiryaev, A.N. (1995) Optimization of the flow of dividends, *Russian Math. Surveys* **50**, 257–277.
- [30] Kyprianou, A.E. (2006) *Introductory lectures on fluctuations of Lévy processes with applications*, Springer.

- [31] Kyprianou, A.E and Palmowski, Z. (2007) Distributional study of De Finetti's dividend problem for a general Lévy insurance risk process, *Journal of Applied Probability* **44**(2), 428-443.
- [32] Lefèvre, C. and Loisel, S. (2008) On Finite-Time Ruin Probabilities for Classical Risk Models, *Scandinavian Actuarial Journal*, 41–60.
- [33] Loeffen, R. (2008) On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes, *Annals of Applied Probability* **18**(5), 1669-1680.
- [34] Loeffen, R. (2008) An optimal dividends problem with transaction costs for spectrally negative Lévy processes, *Manuscript*.
- [35] Loeffen, R. and Renaud, J. (2009) De Finetti's optimal dividends problem with affine penalty function at ruin, *Manuscript*.
- [36] Müller, A. (1997a) Stop-loss order for portfolios of dependent risks, *Insurance Math. Econom.* **21**, 219–223.
- [37] Müller, A. (1997b) Stochastic orderings generated by integrals: a unified study, *Adv. Appl. Probab.* **29**, 414–428.
- [38] Paulsen, J. (2007) Optimal dividend payments until ruin of diffusion processes when payments are subject to both fixed and proportional costs, *Advances in Applied Probability* **39**, 669–689.
- [39] Picard, P. and Lefèvre, C. (1997) The probability of ruin in finite time with discrete claim size distribution, *Scandinavian Actuarial Journal*, 58–69.
- [40] Rolski, T., Schmidli, H., Schmidt, V. and Teugles, J.L. (1999) *Stochastic processes for insurance and finance*. John Wiley and Sons, Inc., New York.
- [41] Sundt, B. (1999) On multivariate panjer recursions, *ASTIN Bulletin* **29**(1), 29–45.
- [42] Zajic, T (2000) Optimal Dividend Payout under Compound Poisson Income, *Journal of Opt. Theor. and Appl.* **104**, 195-213.
- [43] Zhou, X. (2005) On a classical risk model with a constant dividend barrier, *North American Actuarial Journal* **9**, 1–14.